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Math Review for Derivatives

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Contents

I. Interest Rates and Rates of Return

There are many conventions for quoting interest rates on different financial instruments, and different ways to express rates in doing theoretical and practical calculations. It is essential to understand how rates are being computed and to be able to convert from one convention to another.

Holding period return: The most basic return is given by what you have at the end of your investment period divided by what you invested at the beginning, minus 1.

$$R_{HPR} = (V_1 + c)/V_0 - 1$$

where R_{HPR} = the holding period return,
V_1 = asset price at the end of the holding period,
V_0 = initial asset price (the amount invested)
c = end of period value of any cash payment received (dividend, coupon interest, etc.)

Example: Buy a share of XYZ stock at 50, receive $1.00 dividend, sell in 3 months at 52. This produces a holding period return of

$$R_{HPR} = (V_1 + c)/V_0 - 1 = (52 + 1.00)/50 - 1 = 1.06 - 1 = .0600 = 6.00\%$$

A difficulty: The length of the holding period does not enter the calculation, so it is hard to compare returns for different investment horizons.

Annualizing returns: To make rates comparable for different holding periods, they are normally expressed on an annualized basis. There are several ways to do this:

Simple interest: Assumes you earn interest at the same rate per day for a year, and receive it all at the end.

Compound interest: Assumes you receive the principal and interest at the end of the holding period, then reinvest at the same holding period return. The return compounds as many times as there are holding periods in a year.

Continuous compounding: Treats the return as if it accrues continuously, as a constant rate of growth applied to the initial principal plus all interest accrued so far. If r is the interest rate, an investment of $1 grows to $\$1\ e^{r}$ over one year, and to $\$1\ e^{rt}$ over t years (where t does not have to be an integer).

I. Interest Rates and Rates of Return, p.2

Simple interest

Simple interest assumes you earn the same interest per day for a year, and does the calculation as if it all were received at the end:

Example

Buy a share of XYZ stock at 50, receive $1.00 dividend, sell in 3 months at 52. This produces a holding period return of 6.00%

$$R_{HPR} = (52 + 1.00)/50 - 1 = 1.06 - 1 = .0600 = 6.00\%$$

Annualizing this holding period return at simple interest gives

$$R_{SIMPLE} = 4 \times 6.00 = 24.00\%$$

General Formula

$$R_{SIMPLE} = N \times R_{HPR}$$

where N = number of holding periods per year

If n = the length of the holding period in years,
then N = 1 / n

For example:

If the holding period is 3 calendar days, N = 365 / 3 = 121.7;
if the holding period is 3 trading days, N = (about) 255 / 3 = 85;
if the holding period is 3 months, N = 12 / 3 = 4

Exception:

If the holding period is 3 years, N = 1 / 3. But, compounding is used for periods over 1 year. So for n > 1, the simple rate is given by the nth root of $(1 + R_{HPR}) - 1$, i.e.,

$$R_{SIMPLE} = (1 + R_{HPR})^{1/n} - 1 = (1.06)^{1/3} - 1 = 1.0196 - 1 = 1.96\%$$

I. Interest Rates and Rates of Return, p.3

Compound interest

Compound interest assumes you receive the principal and interest at the end of the holding period, then reinvest at the same holding period return. The return compounds as many times as there are holding periods in a year.

Example

Buy a share of XYZ stock at 50, receive $1.00 dividend, sell in 3 months at 52. This produces a holding period return of 6.00%

$$R_{HPR} = (52 + 1.00)/50 - 1 = 1.06 - 1 = .0600 = 6.00\%$$

Annualizing this holding period return at compound interest gives

$$R_{COMPOUND} = 1.06^{12/3} - 1 = 1.06^{4} - 1 = 1.2625 - 1 = 0.2625 = 26.25\%$$

General Formula

$$R_{COMPOUND} = (1 + R_{HPR})^{N} - 1$$

where N = number of holding periods per year

If n = the length of the holding period (in years),
then N = 1/n

This gives the holding period return that would result from investing over a whole year, with compounding, at the rate of R_{HPR} per period of length n.

I. Interest Rates and Rates of Return, p.4

Annual Percentage Rate (APR) and Effective Annual Rate (EAR)

By law, lenders must tell borrowers the "annual percentage rate" (APR) their interest payments will be based on. This is the number that will be used in the calculations. (APR is treated like a simple interest rate.)

Typically, interest payments are made periodically during a year. For example, a bank may pay interest monthly or daily. The interest payment for an n-day period is calculated as n / 365 of the quoted annual rate R_{APR}.

Treating a single n-day interval as a holding period, we have the holding period return corresponding to a given APR

$$R_{HPR} = n / 365 \ R_{APR}$$

Clearly, the actual total interest paid on funds invested at this rate over multiple n-day holding periods will involve compounding.

For a depositor, the total (compounded) amount that would be accumulated by the end of the year, per dollar invested initially, is known as the Effective Annual Rate (EAR). With compounding every n days, the EAR is

$$1 + R_{EAR} = (1 + R_{HPR})^{365 / n} = (1 + (n / 365) R_{APR})^{365 / n}$$

<u>Example:</u> A quoted 12.00% APR compounded quarterly will translate to an Effective Annual Rate of:

$$R_{EAR} = (1 + 0.12 / 4)^{4} - 1 = 1.03^{4} - 1 = 12.55\%$$

The same 12.00% APR compounded daily gives

$$R_{EAR} = (1 + 0.12 / 365)^{365} - 1 = 1.000329^{365} - 1 = 12.75\%$$

I. Interest Rates and Rates of Return, p.5

Continuous compounding

Continuously compounded rates are conceptually less familiar, but they make many derivatives computations a lot easier.

Continuous compounding treats the return as if it accrues continuously, in the form of a constant rate of growth applied to the entire amount of initial principal plus all accrued interest. If r is the continuously compounded rate, over one year, an investment of \$1 will grow to \$1 e^r, where e = 2.71828, the base of natural logarithms.

General Formula

\$1 invested at the continuously compounded rate r over t years grows to

$$\$1\ e^{rt}$$

Note that t need not be an integer, which makes continuous compounding a lot easier than calculations with holding periods.

Thus, if the holding period return for an n-day holding period (t = n / 365) is R_{HPR}, we have

$$e^{rt} = 1 + R_{HPR}$$

Solving for the continuous rate, r, by taking logarithms, gives

$$r = (1 / t) \ln(1 + R_{HPR}),$$

where ln(.) denotes the natural logarithm.

<u>It is important to understand clearly how these conversions work.</u>

Continuous compounding, continued

Examples

1. R_{HPR} = 3.00% for three months corresponds to a continuously compounded annual rate of

$$r = (1 / t)\ \ln(1 + R_{HPR}) = (1 / .25)\ \ln(1.0300)$$

$$= 4\ (.029559) = .1182$$

$$= 11.82\%$$

2. For a 1 year holding period, a 12.00% continuous rate (i.e., r = 0.12) will produce

$$R_{EAR} = e^{r\,(1)} - 1 = e^{0.12} - 1 = 1.1275 - 1 = 12.75\%$$

3. A zero coupon bond is purchased at a price of 85. Two years later it will mature and pay off 100. The continuously compounded yield is given by

$$V_1 / V_0 = 100 / 85 = 1.1765 = (1 + R_{HPR})$$

$$r = (1 / t)\ \ln(1 + R_{HPR}) = (1 / 2)\ \ln(1.1765) = .0813 = 8.13\%$$

4. **(Rule of 69)** If an investment is made at a continuously compounded rate of r, how many years will it take to double? Answer: divide 69 by the interest rate in percent. So at 6.9%, money doubles in 10 years; at 10% it doubles in 6.9 years; at 23% it doubles in 3 years.

Why does this work? Since V_t is double V_0, we have

$$V_t / V_0 = 2 = (1 + R_{HPR}) = e^{rt}$$

$$r = (1 / t)\ \ln(1 + R_{HPR}) = (1 / t)\ \ln(2) = 0.69 / t$$

$$t = 0.69 / r = 69 / r \text{ in } \%$$

For rates compounded over discrete holding periods (like the yield to maturity on a typical bond), a similar "Rule of 72" holds approximately, for moderate interest rates.

I. Interest Rates and Rates of Return, p.7

Present Value or Zero Coupon Bond Price

An equivalent way to express the time value of money relationship embodied in an interest rate is by the **present value** as of date t, of $1 to be paid at a later date T.

Often it is convenient to think of the date t present value for discounting a $1 payment on future date T as **market price for a zero coupon bond** with face value of $1 that will mature on date T. This is also called the **discount factor**.

$$B_{t,T} = 1 / (1 + R_{HPR})$$

where $B_{t,T}$ = the zero coupon bond price as of date t for a bond that will pay $1 on date T

R_{HPR} = the total holding period return over the period from date t to date T

For many purposes, it is very convenient to summarize the time value of money in a single symbol, as the price of the appropriate zero coupon (or "pure discount") bond. This allows us to avoid explicitly dealing with a lot of messy real world details, such as different market conventions for quoting and compounding interest rates, rates that vary over the holding period, and so on.

Regardless of how the interest calculation are done, $B_{t,T}$ expresses the end result in the form that we need, as the dollar value of an investment on date t that would yield one dollar payoff at T. Thus, the value V_t (appropriate price today) for a security that will pay V_T dollars at date T is simply $V_t = B_{t,T}\ V_T$. (Note that this assumes no risk, or at least that there is no risk premium on the investment.)

To turn the relationship around to obtain the **future value** as of T of $1 at date t, we simply take $1 / B_{t,T}$ (which just gives $1 + R_{HPR}$).

Examples:

If the quoted (APR) interest rate is 12%, discount factors for bonds with 6 month (T=1/2), 1 year and 2 year maturities are as follows:

$B_{0,1/2} = 1 / (1 + .12 / 2) = 1 / 1.06 = 0.9434$ (Recall that the APR uses simple
$B_{0,1} = 1 / (1 + .12) = 1 / 1.12 = 0.8929$ interest for short periods but is
$B_{0,2} = 1 / (1 + .12)^2 = 1 / 1.2544 = 0.7972$ compounded for periods greater than 1 year.)

I. Interest Rates and Rates of Return, p.8

Theoretical Yield to Maturity for a Coupon Bond

A normal bond pays regular coupon interest, typically twice a year, plus its face value (reimbursement of principal) at maturity. For a 30 year bond, this amounts to a total of 60 payments. The price of the bond in the market will be equal to the sum of the present values of all of the payments, each one discounted at the interest rate appropriate for its maturity.

The key to fully understanding how any investment actually works is to identify the exact number of dollars that will be paid and the dates on which they will be paid.

The regularity of the payments for a coupon bond makes it useful to compute a single "yield to maturity" (YTM) for the bond. The YTM is the single discount rate that makes the sum of the present values of all of the future cash flows exactly equal to the bond's current market price.

Theoretical Yield to Maturity
Suppose a bond will make cash payments of C_1, C_2, ... , C_T at times t_1, t_2, ... , T , where t_1 is the number of years to the first payment, t_2 is the number of years to the second payment, etc.

The present value of the cash flow stream using the single discount rate r is given by

$$PV = \frac{C_1}{(1+r)^{t_1}} + \frac{C_2}{(1+r)^{t_2}} + \ldots + \frac{C_T}{(1+r)^{T}}$$

The value of r that makes the present value equal the market price is the theoretical yield to maturity.

Market convention, however, is to use a slightly different formula for standard coupon bonds with periodic payments.

I. Interest Rates and Rates of Return, p.9

Yield to Maturity for a Coupon Bond as Quoted in the Market

Market Convention Yield to Maturity

One difference from the theoretical calculation is that the annual "yield" is calculated by finding the rate per coupon period and then annualizing that rate at simple interest, by multiplying it by the number of coupon periods per year. That is, for a typical bond with semiannual coupons, one computes the rate per 6-month period (as in the formula on the previous page) and then doubles it (annualizing at simple interest). In this way, the yield to maturity is treated like an APR.

The second difference is that it is generally necessary to do the calculation in the middle of a coupon period, taking account of both the interest that has already accrued and the fact that the first period is shorter than the others.

Let

P = price of the bond, including accrued interest up to the present

C = annual coupon, paid in equal installments of C/2 semiannually

t_c = number of days before the next coupon date

n = number of days in the current coupon period

F = bond face value

y = yield to maturity

then y is the value that solves the following equation

$$P = \frac{\displaystyle\sum_{s=1}^{T} \frac{C/2}{(1+y/2)^{s-1}} + \frac{F}{(1+y/2)^{T-1}}}{(1+y/2)^{t_c/n}}$$

You don't need to know this equation. But you should understand clearly what the purpose of it is, so if you ever are in a situation where you should apply it, you'll know that you need it.

II. Forward Rates and Forward Prices

An interest rate available in the market for money invested today is known as a **spot** interest rate. Spot rates that are observed at the same time for different maturities T_1 and T_2 , with $T_1 < T_2$, embody the interest that will be earned over the future period between T_1 and T_2 . The interest rate that is embedded in the current spot rates but applies to a future period is called a forward rate.

Example

The 1 year interest rate is 5.00% and the 2 year rate is 6.00%.

The corresponding holding period returns are
$R_{HPR}(1) = 5.00\%$ and $R_{HPR}(2) = (1.06)^2 - 1 = 12.36\%$.

\$1 invested at time 0 would become \$1.05 in one year and \$1.1236 in two years.

Over the second year, our investment will grow from \$1.05 to \$1.1236, so the 1 year forward interest rate is 1.1236 / 1.05 - 1 = 7.01%.

General Formula

Denote the holding period return for money invested from time 0 to T_1 as $R_{HPR}(T_1)$ and the holding period return to T_2 as $R_{HPR}(T_2)$. The forward holding period return for the period T_1 to T_2 , which we write $R^F_{HPR}(T_1,T_2)$, can be obtained from the relation

$$(1 + R_{HPR}(T_1)) \times (1 + R^F_{HPR}(T_1,T_2)) = (1 + R_{HPR}(T_2))$$

which gives

$$R^F_{HPR}(T_1 , T_2) = (1 + R_{HPR}(T_2)) \ / \ (1 + R_{HPR}(T_1)) - 1$$

Convert the forward holding period return into an annualized interest rate in the standard way.

II. Forward Rates and Forward Prices, p.2

Forward rates in terms of ordinary interest rates:
Typically, forward rates are expressed in terms of ordinary interest rates expressed as annualized percents. Let R(T) be the annualized rate for the period from time 0 to time T (i.e., it is the yield on a T year zero coupon bond). The holding period return from now to date T is just $(1 + R_{HPR}(T)) = (1 + R(T))^T$.

Let $R^F(T_1,T_2)$ be the annualized interest rate from T_1 to T_2. The above equation shows it is

$$R^F(T_1, T_2) = [(1 + R(T_2))^{T_2} / (1 + R(T_1))^{T_1}]^{1/(T_2 - T_1)} - 1$$

Example: The 1 year interest rate is 5.00% and the 2 year rate is 6.00%.

Since the holding period for the forward rate is just one year, the forward holding period return and the forward interest rate will be the same:

$$R^F(1,2) = [(1 + R(2))^2 / (1 + R(1))^1]^{1/(2-1)} - 1 = (1.06)^2 / 1.05 - 1$$

$$= 7.01\%$$

II. Forward Rates and Forward Prices, p.3

Forward rates in terms of continuously compounded rates:
Computing forward rates is especially easy with continuously compounded rates, since $(1 + R_{HPR}(T)) = e^{r T}$. The first "General Formula" equation above becomes

$$e^{r_1 T_1} \times e^{r^F_{1,2} \times (T_2 - T_1)} = e^{r_2 T_2}$$

Taking logs and solving gives

$$\ln(e^{r_1 T_1} \times e^{r^F_{1,2} \times (T_2 - T_1)}) = \ln(e^{r_2 T_2})$$

$$r_1 T_1 + r^F_{1,2} \times (T_2 - T_1)) = r_2 T_2$$

The spot rate in the 1st period times the length of the period, plus the forward rate over the future period from T_1 to T_2 times the length of that period, equals the spot rate r_2 for an investment from T_0 (today) to T_2.

The forward rate for the second period is then

$$r^F_{1,2} = (r_2 T_2 - r_1 T_1) / (T_2 - T_1)$$

Example: The 1 year interest rate is 5.00% and the 2 year rate is 6.00%.

The continuously compounded interest rates corresponding to these spot rates are

$r_1 = \ln(1.05) = 0.04879$ and

$r_2 = \ln(1.06) = 0.05827$.

The forward continuous rate is

$$r^F_{1,2} = (r_2 T_2 - r_1 T_1) / (T_2 - T_1) = (0.05827 \times 2 - 0.04879 \times 1) / (2-1)$$

$$= 0.06775,$$

and $$r^F_{1,2} = e^{r^F_{1,2}} - 1 = 7.01\%$$

III. Day Count Conventions

Interest rates for different instruments and in different countries are quoted using several different conventions with respect to the way the number of days in an interest payment period is specified. This is weird, of course, but comes from common practices in the financial markets from before computers.

Suppose the interest rate is quoted as an annualized rate of R percent. We need to be able to compute what the interest payment on a given date will be in dollars. Variations involve differences in the number of days in a year used for annualizing the rate, and in the way the days in a given holding period are determined.

Standard methods are:

Actual / Actual

Actual / 360

30 / 360

Doing interest calculations correctly is obviously important in practice and the formulas are simple. But you don't need to memorize the details for every market. The important thing is to know that there are different day count conventions in use, so that if you ever need to do this kind of calculation, you'll know you need to make some kind of adjustment, and once you have found out the relevant day count convention, how to apply it.

III. Day Count Conventions, p.2

1. Actual / Actual

The interest payment on a given date is

$$\frac{\text{ACTUAL DAYS IN HOLDING PERIOD}}{\text{ACTUAL DAYS IN A PAYMENT PERIOD}} \times R$$

Sample calculation: Coupon payment dates for the 14 percent U.S. Treasury bonds maturing November 2011 were November 15 and May 15, with 1/2 of the annual coupon, 7, paid on each semiannual date.

A bond delivered on March 15, 1996 had 121 (actual) days of interest accrued since the previous payment date. This payment period, from Nov. 15, 1995 - May 15, 1996 contained 182 actual days (1996 was a leap year). The accrued interest would therefore be computed as

$$(121 / 182) \times 7 = 4.6538 \text{ per } \$100 \text{ face value.}$$

Markets where used: Used for U.S. Treasury notes and bonds, French and Australian government bonds. U.K. and Japanese government bonds are almost the same, but they treat all years, even leap years, as having 365 days.

III. Day Count Conventions, p.3

2. Actual / 360

In quoting the rate R, a year is assumed to have 360 days, but payments are based on the actual number of days in the holding period times 1 / 360th of the annual interest rate.

The interest payment on a given date is

$$\frac{\text{ACTUAL DAYS IN HOLDING PERIOD}}{360} \times R$$

Sample calculation: Assume a $1 million LIBOR-based loan at a quoted interest rate of 6.00 percent was outstanding from Nov. 15, 1995 - Mar. 15, 1996. There were 121 actual days in the holding period, so the interest payment would be

(121 / 360) x .0600 x $1 million = $20,166.67

Markets where used: Money market instruments, such as Treasury bills, LIBOR, repurchases, commercial paper, etc., in the U.S. and most other countries. (Note that US T-bills, have the additional complication that the rate R is stated on a "discount basis", as 100 - P_0.)

III. Day Count Conventions, p.4

3. 30 / 360

A year is treated as consisting of 12 months with 30 days each, so a 6 month coupon payment period is assumed to have 180 days.

The interest payment on a given date is

$$\frac{30 \text{ x MONTHS } + \text{ ACTUAL DAYS IN CURRENT MONTH}}{360} \text{ x R}$$

Sample calculation: Assume coupon payment dates for a 14 percent coupon XYZ corporation bond are November 15 and May 15, with 6 / 12 of the annual coupon, 7, paid on each semiannual date.

A bond delivered on March 20, 1996 had 4 months (Nov. 15 to March 15) plus 5 days of accrued interest, i.e., 125 days. The 6 month coupon payment period is assumed to be 180 days. The accrued interest would therefore be computed as

$$((4 \text{ x } 30 + 5) / 180) \text{ x } 7 = 4.8611 \text{ per \$100 face value.}$$

Markets where used: U.S. federal agency debt, U.S. corporate bonds, Eurobonds, Dutch and German government bonds

III. Day Count Conventions, p.5

Trading Days versus Calendar Days

For some financial calculations you should use calendar days (365 or 366 per year). For others you should use "trading" or "business" days (about 255 per year, i.e., excluding weekends and holidays).

The way to think about this is as follows:

For interest calculations, you want to take account of the fact that interest accrues every day, not just when the financial markets are open. In this case a year really is 365 days, on which interest accrues.

For calculations that involve market prices, like annualizing observed stock returns or calculating volatility, you only get an observation when the market is open, which is only about 255 days a year. If you need market prices to do the calculation, treat a year as having (about) 255 days.

The exact number of business days may vary from year to year. The main point is that 365 is way off and a number in the range of 250 – 260 is a lot closer.

<u>One final point, when combining data from different countries:</u> Holidays occur on different days in different countries, so you have to be careful if you want to look at things like correlations that require returns in two markets to be simultaneous. Don't make a mistake by inadvertently comparing the New York stock returns on Tuesday with Wednesday's return in the U.K. because the London market was closed on Tuesday for the Queen's Birthday.

IV. Means, Variances, Standard Deviations, Correlations

Derivatives are used to manage risk. In modern financial theory, risk is a probabilistic concept. Understanding derivatives requires being comfortable with some basic probability theory. This section reviews the mathematics of means, variances, standard deviations and correlations that we will use in analyzing derivatives.

We will use the following notation:

Let **x** and **y** represent random variables (like stock returns or future interest rates)

The mean of **x** : $E[x] = \bar{x}$

The variance of **x** : $VAR[x] = \sigma_x^2$

The standard deviation of **x** : $STD[x] = \sigma_x$

The covariance between **x** and **y** : $COV[\mathbf{x}, \mathbf{y}]$

The correlation between **x** and **y** : $CORR[\mathbf{x}, \mathbf{y}] = \rho_{xy}$

These refer to theoretical (true) values. If we are just starting with raw data, these statistics have to be estimated, and the estimates will always have estimation errors in them.

IV. Means, Variances, Standard Deviations, Correlations

Discrete Distributions

When there are only a finite number of possible outcomes for some random variable **x**, such as in flipping a coin or rolling dice, the probability distribution is just the set of probabilities for the possible **x** values:

$$\text{Prob}(\, x_j \,) = P_j \qquad \text{for } j = 1, \; \ldots \; , J$$

In that case,

$$E[\, x \,] \; = \; \bar{x} \; = \; \sum_{j=1}^{J} x_j \, P_j$$

$$VAR[\, x \,] \; = \; \sigma_x^2 \; = \; \sum_{j=1}^{J} (\, x_j - \bar{x} \,)^2 \, P_j$$

Example:

Suppose today's stock price is S_0 = 100 and the stock price has a binomial distribution (only two possible outcomes): S_1 = 150 with probability 0.6 and S_1 = 50 with probability 0.4.

1. $E[\, S_1 \,] = 0.6 \text{ x } 150 + 0.4 \text{ x } 50 = 110$

2. $VAR[\, S_1 \,] = 0.6 \text{ x } (150 - 110)^2 + 0.4 \text{ x } (50 - 110)^2$

   $= 0.6 \text{ x } 1600 + 0.4 \text{ x } 3600 \; = \; 960 + 1440$

   $= 2400$

3. $STD\, [\, S_1 \,] = \sqrt{2400} \; = \; 49.0$

Note that in this case, we are saying that we know the true probabilities.

IV. Means, Variances, Standard Deviations, Correlations

Estimating Means, Variances, and Standard Deviations

Many calculations needed for derivatives involve these concepts. But, unlike coin flipping, we don't know the probabilities. We have to estimate the values we need from past observations.

Suppose we have computed a set of (holding period) returns, $\{R_1, R_2, ..., R_T\}$ over T time periods. We can estimate the

Sample Mean:

$$\bar{R} = \frac{\sum_{t=1}^{T} R_t}{T}$$

Sample Variance:

$$\sigma^2 = \frac{\sum_{t=1}^{T} (R_t - \bar{R})^2}{T-1}$$

Note that T-1 is used in the denominator of the variance equation because computing the mean from the same data set effectively uses up the information in one observation.

Standard Deviation: $\quad \mathrm{STD}[R] = \sigma = \sqrt{\sigma^2}$

Example Over 4 weeks, XYZ stock has had returns of +5.0%, -5.0%, 0, and +8.0%.

Mean = (5.0 + (-5.0) + 0.0 + 8.0) / 4 = 2.0%

Variance = $[(5.0 - 2.0)^2 + (-5.0 - 2.0)^2 + (0.0 - 2.0)^2 + (8.0 - 2.0)^2] / 3$

$= (9.0 + 49.0 + 4.0 + 36.0) / 3 = 32.67 (\%)^2$

Standard deviation = $\sqrt{32.67}$ = 5.72 %

Note that these are the exact mean, variance and standard deviation for this sample of data. They are estimates if you want to assume the returns for other dates come from this same probability distribution.

IV. Means, Variances, Standard Deviations, Correlations

Annualizing Variances and Standard Deviations

For a returns process like what is typically assumed for returns on stocks and most other assets, the random component of return is independent from one period to the next, even if the periods are really short (like 1 second). If the variance per period is $\boldsymbol{\sigma}^2$, the total variance over N periods is $N\,\boldsymbol{\sigma}^2$ and the standard deviation over an N period horizon is

$$\sqrt{N\sigma^2} = \sigma\sqrt{N}$$

If N is the number of periods in a year, then

Annual variance:

$$\sigma^2_{annual} = N\ \sigma^2_{period}$$

Annual standard deviation:

$$\sigma_{annual} = \sqrt{N\ \sigma^2_{period}} = \sqrt{N}\ \sigma_{period}$$

IV. Means, Variances, Standard Deviations, Correlations

Volatility: Volatility is the standard deviation of the continuously compounded (i.e., logarithmic) return on some asset, expressed as an annualized rate. The calculation is the same as annualized standard deviation, for returns defined as $r_t = \ln(1 + R_t)$, where R_t denotes the simple holding period return.

Example:
If the variance of log returns estimated from past data is 2 $(\%)^2$ per trading day (notice the units!), then

standard deviation per trading day = $\sqrt{2.0} = 1.41\%$

annual variance of continuously compounded return = 255 x 2 = 510 $(\%)^2$

volatility = $\sqrt{510} = 22.6\%$

or alternatively, $1.41\sqrt{255} = 22.6\%$

"Square Root of T Rule" If σ is the annualized standard deviation of the return, the standard deviation over some time period of length T is given by $\sigma\sqrt{T}$.

For example, if annual volatility is 20%, the standard deviation of return over a 3 month period (1/4 year) is

$$\sigma_{3\text{ month}} = \sigma\sqrt{1/4} = 20 \times 1/2 = 10\%$$

The square root of T rule is a very important property of the way risk behaves as a function of the time horizon: risk goes up with the square root of the length of the holding period.

BE SURE YOU KNOW AND UNDERSTAND THE SQUARE ROOT OF T RULE!

COVARIANCES AND CORRELATIONS

Covariance: Suppose we have two sets of returns, for assets A and B, denoted $R_{A\,t}$ and $R_{B\,t}$. Estimate their covariance by

$$COV\,[R_A\ ,\ R_B\ ] = \frac{\sum_{t=1}^{T} (\,R_{A\,t} - \bar{R}_A\,)(\,R_{B\,t} - \bar{R}_B\,)}{T - 1}$$

Note that, as with variance, T-1 is used in the denominator when a covariance is computed using deviations from the sample means.

Correlation: Correlation is a standardized covariance, that is transformed to lie between $\rho = -1$ and $\rho = +1$ (ρ is the Greek letter rho):

$$\rho_{AB} = COV\,[R_A\,,\,R_B]\ /\ (\sigma_A\ \sigma_B)$$

Example: For the last 4 weeks, XYZ and ABC had returns of

XYZ:	+5.0,	-5.0,	0.0,	+8.0	percent,
ABC:	+6.0,	-3.0,	-1.0,	12.0	percent.

The same types of calculations as above show the mean, variance and standard deviation of ABC to be +3.5%, 47.0 ($\%^2$), and 6.86%, respectively.

$COV[R_{XYZ},R_{ABC}] = [(3.0 \times 2.5) + (-7.0 \times -6.5) + (-2.0 \times -4.5) + (6.0 \times 8.5)]\ /\ 3$

$= 37.67\ (\%^2)$

CORRELATION: $\rho_{XYZ,ABC} = 37.67\ /\ (5.72 \times 6.86) = 0.96$

[Note: The (old) Excel spreadsheet covariance function COVAR incorrectly puts T instead of T-1 in the denominator for the calculation. This means that in Excel, CORREL(X,Y) does not equal COVAR(X,Y) / (STDEV(X) * STDEV(Y)), even though it must, mathematically.] More recent Excel has introduced functions ending in .S and .P, such as VAR.S and VAR.P, with .S meaning Sample (divide by T-1) and .P meaning Population (divide by T). Use .S when calculating with data, and .P when you know the true probabilities.

V. Probability Formulas for Portfolios and Hedges

Portfolios and hedged positions involve combinations of risky and riskless assets. The resulting mean, variance, and standard deviation for the whole portfolio are governed by the following formulas.

Linear functions of a random variable

If $\mathbf{y} = a\mathbf{x} + b$, where $\mathbf{x}$ is a random variable and a and b are constants,

$$E[\mathbf{y}] = E[a\mathbf{x} + b] = a\,E[\mathbf{x}] + b$$

- The expected value of the sum of a random variable and a constant is the expected value of the random variable, plus the constant.
- The expected value of a variable multiplied by a constant is the constant times the expected value of the variable.

$$\mathrm{Var}[y] = \mathrm{Var}[ax + b] = a^2\,\mathrm{Var}[x] = a^2\sigma_x^2$$

$$\mathrm{Std\,Dev}[y] = \sqrt{\mathrm{Var}[ax+b]} = a\,\sigma_x$$

- The variance of a random variable plus a constant is the same as the variance of the random variable alone; adding a constant does not change variance.
- The variance of a random variable multiplied by a constant is the variance of the random variable times the <u>square</u> of the constant.

<u>Example:</u>
The annual mean and standard deviation of the return on XYZ stock are 15.0% and 30.0%, respectively, and the riskless rate is 5.0%.

The expected return and standard deviation for a position Y that is 30% invested in stock and 70% in the riskless asset will be $E[0.30\,R_{XYZ} + 0.70\,R_F]$

$$E[R_y] = 0.30\,E[R_{xyz}] + 0.70\,(5.0) = 4.5 + 3.5 = 8.0\%$$

$$\boldsymbol{\sigma_y} = 0.30\,\boldsymbol{\sigma_{xyz}} = 0.30\,(30.0) = 9.0\%$$

Combinations of Random Variables

If x and y are both random variables, a weighted sum of the form
$z = w_1 x + w_2 y$ (as in a portfolio or a hedged position) has the following mean and variance:

$$E[z] = E[w_1 x + w_2 y] = w_1 \bar{x} + w_2 \bar{y}$$

$$\begin{aligned} Var[z] &= Var[w_1 x + w_2 y] \\ &= w_1^2 \sigma_x^2 + w_2^2 \sigma_y^2 + 2 w_1 w_2 \rho \sigma_x \sigma_y \end{aligned}$$

where ρ denotes the correlation coefficient between x and y.

Portfolio example:

Two stocks, ABC and XYZ have the following return and risk characteristics:

$E[R_{ABC}] = 10.0\%$ $\sigma_{ABC} = 25.0\%$ $\rho = 0.6$

$E[R_{XYZ}] = 15.0\%$ $\sigma_{XYZ} = 40.0\%$

A portfolio P with half of the funds invested in ABC and half in XYZ would have

$$E[R_P] = 0.5 \times (10.0) + 0.5 \times (15.0) = 12.5\%$$

$$\begin{aligned} Var[R_P] &= 0.5^2 \times 25.0^2 + 0.5^2 \times 40.0^2 + 2 \times 0.5 \times 0.5 \times 0.6 \times 25.0 \times 40.0 \\ &= 156.25 + 400.00 + 300.00 \\ &= 856.25 \end{aligned}$$

$$\sigma_P = \sqrt{856.25} = 29.3\%$$

V. Probability Formulas for Portfolios and Hedges, p.3

Hedging example

Gold and gold futures price changes have the following standard deviations and correlation:

Gold: **Std Dev [ΔP] = σ_P = 11.0**

Gold futures: **Std Dev [ΔF] = σ_F = 11.3**

correlation: ρ **= 0.96**

A one-for-one hedged position (that is, w_P = 1.0 and w_F = - 1.0) has variance and standard deviation of

$$\sigma_H^2 = \sigma_P^2 + \sigma_F^2 - 2 \rho \sigma_P \sigma_F$$

$$= 121.00 + 127.69 - 238.66 = 10.03$$

$$\sigma_H = \sqrt{10.03} = 3.17\%$$

